A Short Proof of the Schröder-Simpson Theorem

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Continuous Valuations

Definition

Continuous valuation ν = map $\nu \colon \mathcal{O}(X) \to \overline{\mathbb{R}}^+$ such that:

$$\begin{array}{rcll}
\nu(\emptyset) & = & 0 & \text{(strictness)} \\
U \subseteq V & \Rightarrow & \nu(U) \leq \nu(V) & \text{(monotonicity)} \\
\nu(U \cup V) + \nu(U \cap V) & = & \nu(U) + \nu(V) & \text{(modularity)} \\
\nu(\bigcup^{\uparrow}_{i \in I} U_i) & = & \sup^{\uparrow}_{i \in I} \nu(U_i) & \text{(continuity)}
\end{array}$$

- Similar to measures, except give weights to opens.
- Needed in semantics of programming languages [JonesPlotkin89]
- Under assumptions on X, cont. valuation=regular measure [KeimelLawson05].

The Weak Topology

Let $\overline{\mathbb{R}}^+_\sigma = \overline{\mathbb{R}}^+$ with Scott topology (opens $= (t, +\infty]$).

Definition

Let $\mathbf{V}(X)$ = set of continuous valuations of X,
with the weak topology,
coarsest making $\nu \mapsto \int_x h(x) d\nu$ continuous, for every continuous h.
Subbasic opens:

$$[h > r] = \{\nu \in \mathbf{V}(X) \mid \int_x h(x) d\nu > r\}$$

- As the usual weak topology, except test functions h are continuous to $\overline{\mathbb{R}}^+_\sigma$ (=lsc).

A Riesz Representation Theorem

Write $\mathcal{C}(Y)$ = space of continuous maps $[Y \to \overline{\mathbb{R}}^+]$ (=lsc)
Y^* = subspace of linear continuous maps

Theorem (Kirch93, Tix95)

The following is a homeomorphism:

$$\mathcal{C}(X)^* \underset{\lambda h \in \mathcal{C}(X) \cdot \int_x h(x) d\nu \;\longleftarrow\!\shortmid\; \nu}{\overset{G \;\longmapsto\; \lambda U \in \mathcal{O}(X) \cdot G(\chi_U)}{\rightleftarrows}} \mathbf{V}(X)$$

- No condition on X at all
- Easy proof

The Schröder-Simpson Representation Theorem

Write $\mathcal{C}(Y)$ = space of continuous maps $[Y \to \overline{\mathbb{R}}^+]$ (=lsc)
Y^* = subspace of linear continous maps

Theorem (SchröderSimpson05)

The following is an isomorphism of cones:

$$\mathcal{C}(X) \xrightarrow[\longleftarrow]{h \longmapsto \lambda\nu\in\mathbf{V}(X)\cdot\int_{x\in X} h(x)d\nu} \mathbf{V}(X)^*$$

- No condition on X at all
- Schröder and Simpson's proof [Simpson09] was elaborate

Topological Cones

Definition

A cone $(C, +, .)$ is as a vector space,
except scalar product $r.c$ is with non-negative reals r.
Topological cone $=$
cone with $+ \colon C \times C \to C$, $. \colon \mathbb{R}^+_\sigma \times C \to C$ continuous.

- Example: $\mathbf{V}(X)$, weak topology
- Non-example: $\mathcal{C}(X)$, Scott topology (unless X core-compact)
  . . . product not *jointly* continuous
- Linear maps: $\psi(\sum_i r_i \nu_i) = \sum_i r_i \psi(\nu_i)$ for $r_i \geq 0$

The "obvious" way to a proof

$$\mathcal{C}(X) \underset{?}{\overset{h \longmapsto \lambda\nu\in\mathbf{V}(X)\cdot\int_{x\in X} h(x)d\nu}{\rightleftarrows}} \mathbf{V}(X)^*$$

- Main task: given $\psi \colon \mathbf{V}(X) \to \overline{\mathbb{R}}^+$ linear continuous, find $h \in \mathcal{C}(X)$ such that $\psi(\nu) = \int_{x\in X} h(x)d\nu$ for every ν
- If h exists, then $\psi(\delta_x) = \int_x h(x)d\delta_x = h(x)$

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- If h exists, then $\psi(\delta_x) = \int_x h(x)d\delta_x = h(x)$
- So only one possible choice: $h(x) = \psi(\delta_x)$
- Need to show

$$\int_x \psi(\delta_x)d\nu = \psi(\nu) \tag{1}$$

for every $\nu \in \mathbf{V}(X)$ . . . really hard.

The "obvious" way to a proof (2)

Let's try and show (1):

$$\int_X \psi(\delta_x) d\nu = \psi(\nu)$$

- Obvious if $\nu = \sum_{i=1}^{m} a_i \delta_{x_i}$ (simple)
- Easy for ν quasi-simple (directed sup of simple valuations), by continuity
- **Problem**: not all continuous valuations are quasi-simple.

Previous proofs

- [SchröderSimpson 05], [Simpson 09]:
  elaborate series of deep results and finely bounding inequalities
- [Keimel 12]: imitates classical proof of similar, measure-theoretic theorem;
  develops nice theory of quasi-uniform separation in (quasi-uniform) cones.

Previous proofs

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  develops nice theory of quasi-uniform separation in (quasi-uniform) cones.
- **Our proof**: look at the weak open $\psi^{-1}(1, +\infty]$
  ...must be of the form $\bigcup_{i \in I} \bigcap_{j=1}^{n} [h_{ij} > r_{ij}]$
  ...we can take $r_{ij} = 1$;
  ...use **Lemma 1** to eliminate unions
  ...use **Lemma 2** to eliminate intersections
  ...so $\psi^{-1}(1, +\infty] = [h > 1]$: this is the right h.

Proof, Lemma 1

A surprising observation (mostly due to [Keimel12])

Lemma 1 (Linear cont. ψ cannot tell between sup $\int$ and $\int$ sup)

Let $h_i \in \mathcal{C}(X)$, $i \in I$. TFAE:
(1) $\psi(\nu) \geq \sup_i \int_x h_i(x) d\nu$ for every ν
(2) $\psi(\nu) \geq \int_x \sup_i h_i(x) d\nu$ for every ν

Note: $\sup_i$ not directed, $\sup_i \int_x h_i(x) d\nu \neq \int_x \sup_i h_i(x) d\nu$.
Proof (1/2). Only $1 \Rightarrow 2$ is non-trivial.

- Trick 1. Every h_i directed sup of step functions
  so can assume each h_i step, $= \sum_k a_{ik} \chi U_{ik}$
- Trick 2. $\sup_{i \in I} = \sup^{\uparrow}_{J \text{ finite } \subseteq I} \sup_{i \in J}$
  by continuity of $\int$, can assume I finite.

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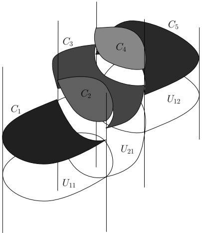
Proof (2/2). Assume $h_i = \sum_k a_{ik} \chi_{U_{ik}}$, $i \in I$ finite.

- Let C_m be atoms of Boolean algebra of sets generated by U_{ik}
- On each C_m, h_i constant $= a_{im}$
  so $\int_x \sup_i h_i(x) d\nu_{|C_m} = \max_i a_{im} \nu(C_m)$
  $= \sup_i \int_x h_i(x) d\nu_{|C_m}$
- By 1, $\psi(\nu_{|C_m}) \geq \int_x \sup_i h_i(x) d\nu_{|C_m}$
- Since ψ additive and $\nu = \sum_m \nu_{|C_m}$, 2 follows.



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Lemma 1, Corollary

$\psi^{-1}(1,+\infty] = \bigcup_{i\in I}[h_i > 1] \Rightarrow \psi^{-1}(1,+\infty] = [h > 1]$ with $h = \sup_i h_i$.

Proof. $\subseteq$ obvious. For $\supseteq$:

- $[h_i > 1] \subseteq \psi^{-1}(1,+\infty]$ implies $\psi(\nu) \geq \int_x h_i(x)d\nu$ for all ν, i
- So 1 holds, hence 2: $\psi(\nu) \geq \int_x h(x)d\nu$.
- If $\nu \in [h > 1]$, $\psi(\nu) \geq \int_x h(x)d\nu > 1$, so $\nu \in \psi^{-1}(1,+\infty]$.

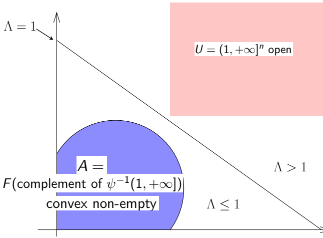
Proof, Lemma 2

Lemma 2

If $\bigcap_{j=1}^{n}[h_j > 1] \subseteq \psi^{-1}(1,+\infty]$,
there is $h \in \mathcal{C}(X)$ such that $\bigcap_{j=1}^{n}[h_j > 1] \subseteq [h > 1] \subseteq \psi^{-1}(1,+\infty]$

Proof.

- Let $F(\nu) = (\int_x h_1(x)d\nu, \cdots, \int_x h_n(x)d\nu)$ in $\overline{\mathbb{R}}^{+n}$
- Separate by $\Lambda(\vec{c}) = \sum_j \gamma_j c_j$ linear continuous (e.g. [Keimel 2006])
- Let $h = \sum_j \gamma_j h_j$. Note $\int_x h(x)d\nu = \sum_j \gamma_j \int_x h_j(x)d\nu = \Lambda(F(\nu))$ Inequalities follows.



End of proof

Let $\psi \in \mathbf{V}(X)^*$, write $\psi^{-1}(1, +\infty]$ as $\bigcup_{i \in I} \bigcap_{j=1}^{n_i} [h_{ij} > 1]$.

Lemma 2 (recap)

There is $h_i \in \mathcal{C}(X)$ s.t. $\bigcap_{j=1}^{n_i} [h_{ij} > 1] \subseteq [h_i > 1] \subseteq \psi^{-1}(1, +\infty]$

So $\psi^{-1}(1, +\infty] = \bigcup_{i \in I} [h_i > 1]$

End of proof

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Lemma 1, Corollary

Then $\psi^{-1}(1, +\infty] = [h > 1]$ with $h = \sup_i h_i$.

For all ν, t, $\psi(\nu) > t$ iff $\nu/t \in \psi^{-1}(1, +\infty]$
iff $\nu/t \in [h > 1]$ iff $\int_x h(x) d\nu > t$.

So $\psi(\nu) = \int_x h(x) d\nu$. $\square$

Recap

Given $\psi \in \mathbf{V}(X)^*$,
there is an $h \in \mathcal{C}(X)$ such that $\psi(\nu) = \int_X h(x) d\nu$ (1)

This h derived from the shape of weak opens $\bigcup_{i \in I} \bigcap_{j=1}^{n} [h_{ij} > r_{ij}]$
... and taking $\nu = \delta_x$ in (1) implies $h(x) = \psi(\delta_x)$.

Recap

We have proved:

Theorem (SchröderSimpson05)

The following is an isomorphism of cones:

$$\mathcal{C}(X) \underset{\lambda x \cdot \psi(\delta_x) \longleftarrow \psi}{\overset{h \longmapsto \lambda \nu \in \mathbf{V}(X) \cdot \int_{x \in X} h(x) d\nu}{\rightleftarrows}} \mathbf{V}(X)^*$$

Notes:

- $\mathbf{V}(X)^* \cong \mathcal{C}(X)$ (here) + $\mathbf{V}(X) \cong \mathcal{C}(X)^*$ (Riesz-Kirch-Tix)
  $\Rightarrow \mathbf{V}(X)$ and $\mathcal{C}(X)$ are dual cones ($\mathcal{C}(X)$ with weak* topology)
- No assumption needed on X.
- Short proof
- Questions?