A Few Pearls in the Theory of Quasi-Metric Spaces

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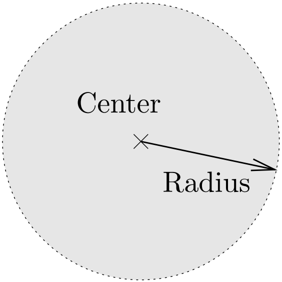
Summer Topology Conference — July 23-26, 2013

Outline

In memory of Paweł Waszkiewicz (1973-2011†).

Outline

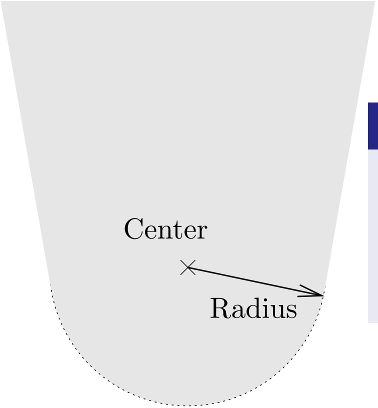
Metric Spaces



Definition (Metric)

- $x = y \Leftrightarrow d(x,y) = 0$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$

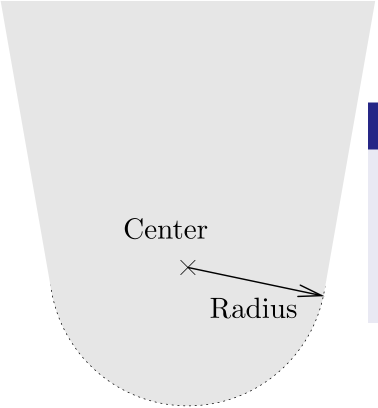
Quasi-Metric Spaces



Definition (Quasi-Metric)

- $x = y \Leftrightarrow d(x, y) = 0$
- ~~$d(x, y) = d(y, x)$~~
- $d(x, y) \leq d(x, z) + d(z, y)$

Hemi-Metric Spaces



Definition (Hemi-Metric)

- $x = y \Rightarrow d(x, y) = 0$
- ~~$d(x, y) = d(y, x)$~~
- $d(x, y) \leq d(x, z) + d(z, y)$

Goals of this Talk

1. Quasi-, Hemi-Metrics a Natural Extension of Metrics
2. Most Classical Theorems Adapt

   ... proved very recently.
3. Formal Balls!

Quasi-Metrics are Natural [Lawvere73]

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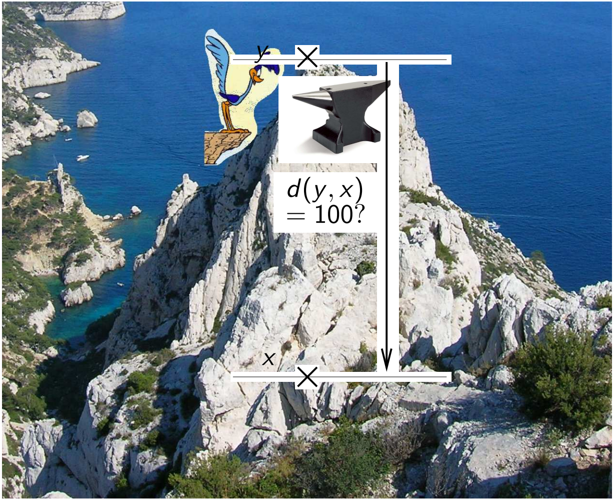
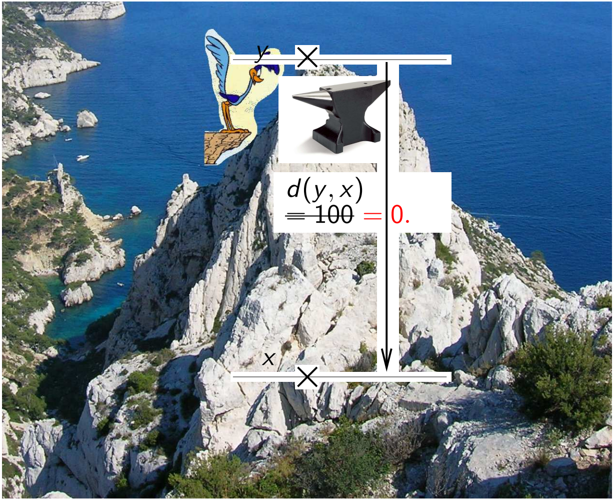


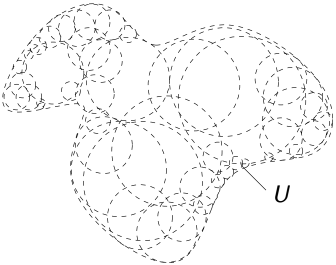
Quasi-Metrics are Natural [Lawvere73]



Outline

The Open Ball Topology

As in the symmetric case, define:



Definition (Open Ball Topology)

An open U is a union of open balls.

The Open Ball Topology

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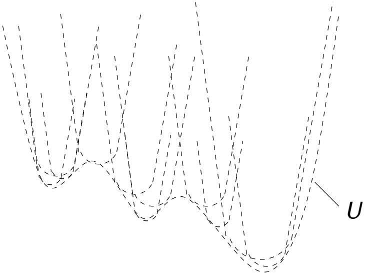


Definition (Open Ball Topology)

An open U is a union of open balls.

. . . but open balls are stranger.

Note: there are more relevant topologies, see later.

The Specialization Quasi-Ordering

Definition ($\leq$)

Let $x \leq y$ iff (equivalently):

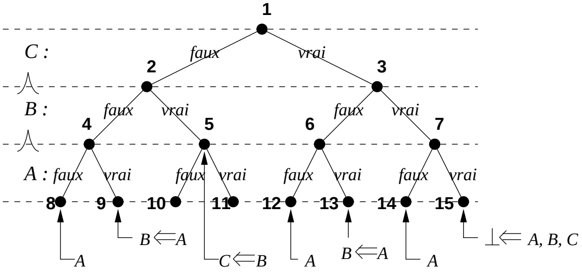
- every open containing x also contains y
- $d(x, y) = 0$.

This would be trivial in the symmetric case.

Example: $\mathrm{d}_{\mathbb{R}}(x, y) = \max(x - y, 0)$ on $\mathbb{R}$.
Then $\leq$ is the usual ordering.

Excuse Me for Turning Everything Upside-Down. . .

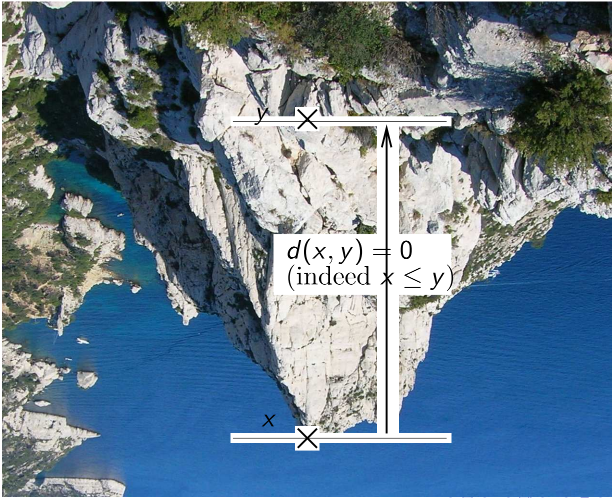
. . . but I'm a computer scientist. To me, trees look like this:



with the root on top, and the leaves at the bottom.

Excuse Me for Turning Everything Upside-Down. . .

. . . but you should really look at hills this way:



Symmetrization

Definition (d^{sym})

If d is a quasi-metric, then

$$d^{sym}(x,y) = \max(d(x,y), \underbrace{d(y,x)}_{d^{op}(x,y)})$$

is a metric.

Example: $\mathrm{d}_{\mathbb{R}}^{sym}(x,y) = |x-y|$ on $\mathbb{R}$.

Motto: A quasi-metric d describes

- a metric d^{sym}
- a partial ordering $\leq$ $\quad (x \leq y \Leftrightarrow d(x,y) = 0)$
- and possibly more.

Pearl 1: Wilson's Theorem

Remember the following classic?

Theorem (Urysohn-Tychonoff, Early 20th Century)

For countably-based spaces, metrizability ⇔ regular Hausdorff.

Proof: hard. □

Pearl 1: Wilson's Theorem

Remember the following classic?

Theorem (Urysohn-Tychonoff, Early 20th Century)

For countably-based spaces, metrizability ⇔ regular Hausdorff.

Proof: hard. □

We have the much simpler:

Theorem (Wilson31)

For countably-based spaces, hemi-metrizability ⇔ TRUE.

Proof: let $(U_n)_{n \in \mathbb{N}}$ be countable base.
Define $d_n(x, y) = 1$ iff $x \in U_n$ and $y \notin U_n$; 0 otherwise.
Together $(d_n)_{n \in \mathbb{N}}$ define the original topology.
Then let $d(x, y) = \sup_{n \in \mathbb{N}} \frac{1}{2^n} d_n(x, y)$. □

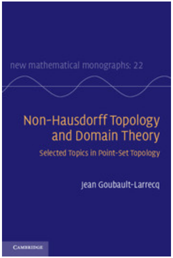
Outline

Completeness

- Completeness is an important property of metric spaces.
- Many generalizations available:
  - Čech-completeness
  - Choquet-completeness
  - Dieudonné-completeness
  - Rudin-completeness
  - Smyth-completeness
  - Yoneda-completeness
  - ...
- I was looking for a unifying notion.
- I failed, but Smyth [Smyth88] and Yoneda [BvBR98] are the two most important for quasi-metric spaces.

A Shameless Ad

Most of this in Chapter 5 of:

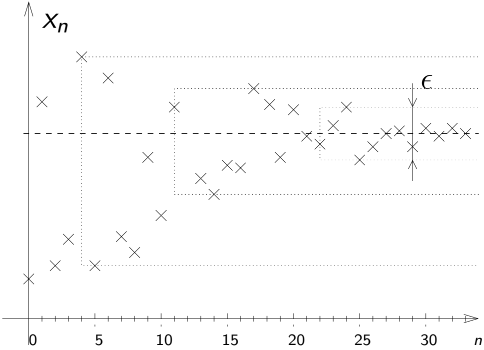


. . . a book on topology (mostly non-Hausdorff) with a view to domain theory (but not only).

Completeness in the Symmetric Case

Definition

A metric space is complete $\Leftrightarrow$ every Cauchy net has a limit.



Definition (Cauchy)

$\forall \epsilon > 0$,
for $i \leq j$ large enough,
$d(x_i, x_j) < \epsilon$

i.e.,
$\limsup_{i \leq j} d(x_i, x_j) = 0$

Basic Results in the Symmetric Case

The following are complete/preserve completeness:

- $\mathbb{R}^{sym}$ (i.e., with $d_{\mathbb{R}}^{sym}(x, y) = |x - y|$)
- every compact metric space
- closed subspaces
- arbitrary coproducts
- countable topological products
- categorical products (sup metric)
- function spaces (all maps/u.cont./c-Lipschitz maps)

Complete Quasi-Metric Spaces

For quasi-metric spaces, two proposals:

Definition (Smyth-c. [Smyth88])	Definition (Yoneda-c. [BvBR98])
Every Cauchy net has a d^{op}-limit	Every Cauchy net has a d-limit

Smyth-complete:

- complete metric spaces
- ~~$\mathbb{R}$~~, ~~$\mathbb{R} \cup \{+\infty\}$~~, $[a, b]$
- symcompact spaces
  i.e., X^{sym} compact
- finite products
- all coproducts
- ~~function spaces~~

. . . with $\mathrm{d}_{\mathbb{R}}(x, y) = \max(x - y, 0)$

Yoneda-complete:

- complete metric spaces
- ~~$\mathbb{R}$~~, $\mathbb{R} \cup \{+\infty\}$, $[a, b]$
- Smyth-complete spaces
  e.g., symcompact spaces
- categ./countable products
- all coproducts
- function spaces (all/c-Lip.)

d-Limits

Used in the less demanding Yoneda-completeness:

Definition (Imitated from definition of directed sup)

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy net
x is a d-limit $\Leftrightarrow \forall y,\ d(x,y) = \limsup_n d(x_n, y)$.

Example: if d metric, d-limit=ordinary limit.

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Example: given ordering $\leq$, $d_{\leq}(x, y) = \begin{cases} 0 & \text{if } x \leq y \\ 1 & \text{else} \end{cases}$:
Cauchy=eventually monotone, d-limit=sup.
In this case, Yoneda-complete=dcpo.

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In this case, Yoneda-complete=dcpo.

Warning: in general, d-limits are not limits (wrt. open ball topol.—need generalization of Scott topology [BvBR98]).

d^{op}-Limits

Used for the stronger notion of Smyth-completeness.
Easier to understand topologically:

Fact

Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy net in X.
Its d^{op}-limit (if any) is its ordinary limit in X^{sym} (if any).

Is there an alternate/more elegant characterizations of these notions of completeness? What do they mean?

Outline

Formal Balls

- Introduced by [WeihrauchSchneider81]
- Characterize completeness through domain theory [EdalatHeckmann98]

  . . . for metric spaces
- A natural idea:
  - Start all over again,
  - look for new relevant definitions of completeness

    . . . this time for quasi-metric spaces,
  - based on formal balls.

Formal Balls

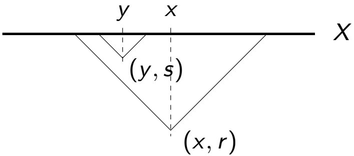
Definition

A formal ball is a pair (x, r), $x \in X$, $r \in \mathbb{R}^+$.

The poset $\mathbf{B}(X)$ of formal balls is ordered by

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) \leq r - s$$

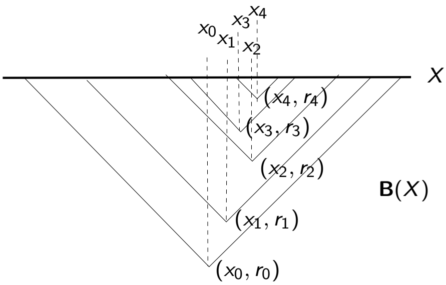
(Not reverse inclusion of corresponding closed balls)



A Theorem by Edalat and Heckmann

Theorem (EdalatHeckmann98)

Let X be metric. X complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo.



Pearl 2: the Kostanek-Waszkiewicz Theorem

Let us generalize to quasi-metric spaces.
How about defining completeness as follows?

Definition (Proposal)

Let X be quasi-metric. X complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo.

Why not, but. . .

Pearl 2: the Kostanek-Waszkiewicz Theorem

Let us generalize to quasi-metric spaces.
How about defining completeness as follows?

Definition (Proposal)

~~Let X be quasi-metric. X complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo.~~

Why not, but... this is a theorem:

Theorem (Kostanek-Waszkiewicz10)

Let X be quasi-metric. X Yoneda-complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo.

Moreover, given chain of formal balls $(x_n, r_n)_{n\in\mathbb{N}}$, with sup (x, r):

- $r = \inf_{n\in\mathbb{N}} r_n$,
- $(x_n)_{n\in\mathbb{N}}$ is Cauchy,
- x is the d-limit of $(x_n)_{n\in\mathbb{N}}$.

Cauchy-weightable Nets

If $(x_n, r_n)_n$ directed family of formal balls and $\inf_n r_n = 0$ then $(x_n)_n$ is Cauchy-weightable:

$$d(x_m, x_n) \quad \leq \quad r_m - r_n \qquad \text{if } m \leq n$$

Lemma

Every Cauchy-weightable net is (forward) Cauchy: for every $\epsilon > 0$,

$$d(x_m, x_n) \quad < \quad \epsilon \qquad \text{for } m \leq n \text{ large enough}$$

Conversely, not all Cauchy nets are Cauchy-weightable
. . . Cauchy-weightability implies $\sum_{n=m}^{+\infty} d(x_n, x_{n+1}) < +\infty$
now take $x_n = (-1)^n/(n+1)$ in $\mathbb{R}$, usual metric.

However. . .

Cauchy $\sim$ Cauchy-weightable

Lemma

TFAE:

1. Every Cauchy net has a d-limit (Yoneda-completeness)
2. Every Cauchy-weightable net has a d-limit

Proof. Every Cauchy net $(x_n)_n$ has a Cauchy-weightable subnet $(x_{\alpha(E)})_{E \text{ finite subset of } ns, \subseteq}$:

- $\alpha(E)$ such that $d(x_m, x_n) < 1/2^{|E|+1}$ for all $m \leq n$ above $\alpha(E)$
  $+ \ \alpha(E) \geq n$ for every $n \in E$ $+ \ \alpha(E) \geq \alpha(E')$ for every $E' \subsetneq E$
- Let $r_{\alpha(E)} = 1/2^{|E|}$ — so $\inf_E r_{\alpha(E)} = 0$
- if $E \subseteq E'$ then $d(x_{\alpha(E)}, x_{\alpha(E')}) \leq 1/2^{|E|+1} \leq r_{\alpha(E)} - r_{\alpha(E')}$
  (distinguish cases $E = E'$ and $|E'| \geq |E| + 1$)

By 2, $(x_{\alpha(E)})_E$ has a d-limit x.
This must also be a d-limit of $(x_n)_n$, since Cauchy. ☐

The Kostanek Waszkiewicz Theorem

Theorem (Kostanek-Waszkiewicz10)

Let X be quasi-metric. X Yoneda-complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo.

Proof ($\Rightarrow$). Let $(x_n, r_n)_n$ directed family of formal balls.
Let $r = \inf_n r_n$: $(x_n, r_n - r)_n$ directed and $\inf_n r_n - r = 0$
so $(x_n)_n$ Cauchy-weighted.
Let $x = d$-limit of $(x_n)_n$.
Now check that $(x, r) = sup_n(x_n, r_n)$ (exercise).

The Kostanek Waszkiewicz Theorem

Theorem (Kostanek-Waszkiewicz10)

Let X be quasi-metric. X Yoneda-complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo.

Proof ($\Leftarrow$). Let $(x_n)_n$ Cauchy-weighted by r_n.
(x_n, r_n) is directed family of formal balls, hence has sup (x, r).
Since $(x_n, r_n) \leq (x, r)$, $r_n \geq r$. So $r = 0$ $(= \inf_n r_n)$.
Check that $x = d$-limit of $(x_n)_n$, i.e., $d(x, y) = \limsup_n d(x_n, y)$:

- Note that $\limsup_n d(x_n, y) = \sup^{\uparrow} d(x_n, y) - r_n$.
- For every y, $d(x_n, y) \leq d(x_n, x) + d(x, y) \leq (r_n - r) + d(x, y)$.
  So $\sup^{\uparrow} d(x_n, y) - r_n \leq d(x, y) - r = d(x, y)$.
- If inequality were strict, let $s = \sup^{\uparrow} d(x_n, y) - r_n < d(x, y)$
  Since $d(x_n, y) \leq r_n + s$, $(x_n, r_n + s) \leq (y, 0)$
  Take sups: $(x, r + s) \leq (y, 0)$, so $d(x, y) \leq r + s = s$,
  contradiction. □

The Continuous Poset of Formal Balls

Let us return to metric spaces for a moment.

Theorem (EdalatHeckmann98)

Let X be metric. X complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo.

The Continuous Poset of Formal Balls

Let us return to metric spaces for a moment.

Theorem (EdalatHeckmann98)

Let X be metric. X complete $\Leftrightarrow$ $\mathbf{B}(X)$ dcpo. Moreover,

- *$\mathbf{B}(X)$ is then a continuous dcpo*
- *and $(x, r) \ll (y, s) \Leftrightarrow d(x, y) < r - s$ (not $\leq$)*

A typical notion from domain theory:

- way-below: $B \ll B'$ iff for every chain $(B_i)_{i \in I}$ such that $B' \leq \sup_i B_i$, then $B \leq B_i$ for some i.
- continuous dcpo = every B is directed sup of all $B_i \ll B$.

Example: $\overline{\mathbb{R}}^+$ ($r \ll s$ iff $r = 0$ or $r < s$)

Pearl 3: the Romaguera-Valero Theorem

Define $\prec$ by: $(x, r) \prec (y, s) \Leftrightarrow d(x, y) < r - s$
How about defining completeness as follows? (X quasi-metric)

Definition (Proposal)

X complete $\Leftrightarrow$ $\mathbf{B}(X)$ continuous dcpo with way-below $\prec$.

Why not, but. . .

Pearl 3: the Romaguera-Valero Theorem

Define $\prec$ by: $(x, r) \prec (y, s) \Leftrightarrow d(x, y) < r - s$
How about defining completeness as follows? (X quasi-metric)

Definition (Proposal)

~~X complete $\Leftrightarrow$ $\mathbf{B}(X)$ continuous dcpo with way-below $\prec$.~~

Why not, but... this is a theorem:

Theorem (Romaguera-Valero10)

X Smyth-complete $\Leftrightarrow$ $\mathbf{B}(X)$ continuous dcpo with way-below $\prec$.

Moreover, given chain of formal balls $(x_n, r_n)_{n \in \mathbb{N}}$, with sup (x, r):

- $r = \inf_{n \in \mathbb{N}} r_n$,
- $(x_n)_{n \in \mathbb{N}}$ is Cauchy,
- x is the d^{op}-limit of $(x_n)_{n \in \mathbb{N}}$, i.e., its limit in X^{sym}.

The Gamut of Completeness Properties So Far

Spaces of formal balls is:

Weaker ↑

Yoneda-complete - - - - - - - - - a dcpo

Smyth-complete - - - - - - - - - a continuous dcpo with $\ll = \prec$

Stronger ↓

Outline

Reminder: Banach's Fixed Point Theorem

Theorem (Banach)

Let X, d complete metric with $\forall x, y,\ d(x, y) < +\infty$.
Every c-Lipschitz map $f : X \to X$ with $c < 1$ has a unique fixed point.

Proof. Sequence of iterates $f^n(x_0)$ is Cauchy,
for any starting point x_0. □

In quasi-metrics, need to correct the following:

- c-Lipschitz does not imply preservation of d-limits

Yoneda-Continuity

Definition (Rutten96 — imitated from Scott-continuity)

Let X, Y Yoneda-complete.
$f : X \to Y$ is Yoneda-continuous
iff uniformly continuous + preserves d-limits of Cauchy nets

- Uniformly continuous $\sim$ monotonic
- Cauchy net $\sim$ directed family
- d-limit of Cauchy net $\sim$ directed sup

Formal Balls to the Rescue (again)

Lemma

Let X, Y Yoneda-complete, $f : X \to Y$ c-Lipschitz.
Define $\mathbf{B}^c(f) : (x, r) \mapsto (f(x), cr)$.

f Yoneda-continuous iff $\mathbf{B}^c(f)$ Scott-continuous.

Proof. By Kostanek-Waszkiewicz,

d-limits of Cauchy-weightable nets
$\equiv$
directed sups of formal balls

□

The Banach-Rutten Fixed Point Theorem

Theorem (Rutten 96)

Let X, d Yoneda-complete with $\forall x, y,\ d(x, y) < +\infty$. Every c-Lipschitz Yoneda-continuous map $f : X \to X$ with $c < 1$ has a unique fixed point.

Proof. Iterates of formal balls $\mathbf{B}^c(f)^n(x_0, r_0)$ form a chain.

...need c-Lipschitz to ensure $(x_0, r_0) \leq \mathbf{B}^c(f)(x_0, r_0)$

for some r_0 large enough $(\geq d(x_0, f(x_0))/(1 - c))$

By Kostanek-Waszkiewicz, has a sup (x, r)

and $\mathbf{B}^c(f)(x, r) = (x, r)$ (Scott-continuity)

So $f(x) = x$. □

The Kleene-Rutten Fixed Point Theorem

Variant, not requiring c-Lipschitzness.

Theorem (Rutten 96)

Let X, d Yoneda-complete with $\forall x, y,\ d(x, y) < +\infty$.
Every Yoneda-continuous map $f : X \to X$ such that $x_0 \leq f(x_0)$ has a least fixed point above x_0

Proof. Iterates of formal balls $\mathbf{B}^c(f)^n(x_0, r_0)$ form a chain

By Kostanek-Waszkiewicz, has a sup (x, r)
and $\mathbf{B}^c(f)(x, r) = (x, r)$ (Scott-continuity)
So $f(x) = x$. □

The Bourbaki-Witt Theorem

Note the pattern:

Existence of a poset-theoretic fixed point on formal balls implies existence of a fixed point in X

Let's use the following:

Theorem (Bourbaki-Witt)

Let Φ be an inductive poset (a dcpo).
Every inflationary map $g : \Phi \to \Phi$ has a fixed point above any given $a \in \Phi$.

Note. Inflationary $= x \leq g(x)$ for every x.
Monotonicity not required.

Pearl 4: the Caristi-Waszkiewicz Theorem

A potential map $\varphi \colon X \to \mathbb{R}^+ =$ (... $\sim$ lower semi-continuity)
for every Cauchy net $(x_n)_n$ with d-limit x, $\varphi(x) \leq \liminf_n \varphi(x_n)$

Theorem (Waszkiewicz 10, generalizing Caristi)

Let X be Yoneda-complete, f an arbitrary map $: X \to X$, φ a potential map such that $\varphi(f(x)) + d(x, f(x)) \leq \varphi(x)$. Then f has a fixed point.

Proof. By Kostanek-Waszkiewicz, $\mathbf{B}(X)$ dcpo.

Since φ potential, $\Phi = \{$formal balls (x, r) with $r \geq \varphi(x)\}$ closed under directed sups, hence dcpo.

Let $f'(x, r) = (f(x), r - \varphi(x) + \varphi(f(x)))$.

Inequality assures well-defined, $f' \colon \Phi \to \Phi$, and inflationary.

By Bourbaki-Witt, $f'(x, r) = (x, r)$ for some (x, r). □

Outline

Alexandroff vs. Scott

Is the open ball topology the right one?

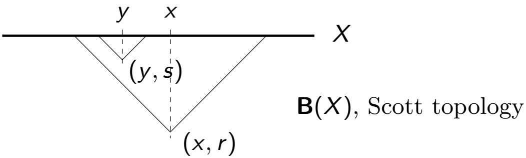
No

- Let $X, \leq$ poset
- Define canonical quasi-metric: $d(x, y) = 0$ iff $x \leq y$, $+\infty$ else
- Then open ball = Alexandroff (open=upward closed)
- The "right" topology is Scott: open=upward closed whose complement is closed under directed sups

The d-Scott Topology

Definition (d-Scott Topology)

= subspace topology
from the inclusion into $\mathbf{B}(X)$ with its Scott topology.



- Same idea as from Lawson's computational models
- Related but not identical with [Bonsangue, van Breugel, Rutten 98]'s generalized Scott topology on X, d
  (identical on algebraic Yoneda-complete spaces)

Properties of the d-Scott Topology

- d-Scott coarser than open ball ($\sim$ Scott vs. Alexandroff)
- d-Scott $=$ open ball if (1) d metric or (2) d Smyth-complete

Proposition

If X, Y Yoneda-complete and f Lipschitz, then
f Yoneda-continuous iff continuous wrt. d-Scott topologies.

Proof. c-Lipschitz f extends to

$$\mathbf{B}^c(f) \colon (x, r) \mapsto (f(x), cr) \text{ on formal balls}$$

…Scott-continuous iff f Y-continuous.

d-Finiteness

Definition (Imitated from domain theory)

x is d-finite iff $d(x, \_)$ Yoneda-continuous from X to $\overline{\mathbb{R}}^{+op}$
iff $d(x, y) = \liminf_n d(x, y_n)$
for every Cauchy $(y_n)_n$ with d-limit y

d-algebraic = every point is d-limit of d-finite points.

- Every point is d-finite if (1) d metric or (2) d Smyth-complete
- In $\mathbb{R} \cup \{+\infty\}$, d-finite points=all except $+\infty$

Pearl 5: the Ali-Akbari Honari Pourmahdian Rezaii Theorem

Theorem

The d-Scott topology of a d-algebraic Yoneda-complete space has a basis of open balls with d-finite centers

...no wonder open ball=d-Scott
if (1) d-metric or (2) d Smyth-complete
since these are cases where every element is d-finite

Theorem (Ali-Akbari, Honari, Pourmahdian, Rezaii 10)

A Yoneda-complete space is Smyth-complete iff all its points are d-finite.

...hence $\mathbb{R} \cup \{+\infty\}$ d-algebraic Y-complete, not S-complete.

The Gamut of Notions of Completeness

Weaker ↑ ... ↓ Stronger

	Spaces of formal balls is:
Yoneda-complete	a dcpo
d-continuous Yoneda-complete	a continuous dcpo
d-algebraic Yoneda-complete	a continuous dcpo with basis (x, r), x d-finite
Smyth-complete	a continuous dcpo with $\ll = \prec$

The Gamut of Notions of Completeness

Spaces of formal balls is:

Weaker ↑

Yoneda-complete		a dcpo
	$\mathbb{R}_\ell, d_\ell$	
d-continuous Yoneda-complete		a continuous dcpo
	$\mathbb{R} \cup \{+\infty\}$ (asym.)	
d-algebraic Yoneda-complete		a continuous dcpo with basis (x, r), x d-finite
Smyth-complete	$[0, 1]$ (asym.)	a continuous dcpo with $\ll = \prec$

↓ Stronger

The Sorgenfrey Line

A famous counterexample in topology (a normal space whose square is not normal).

Definition

$$\mathbb{R}_\ell = \text{reals with } d_\ell(r, s) = \begin{cases} r - s & \text{if } r \geq s \\ 1 & \text{else} \end{cases}$$

("convergence from the right")

- A T_2, non metrizable space (since not second-countable)
- but continuous Yoneda-complete:
  $(x, r) \ll (y, s)$ iff $x > y$ and $x - y < r - s$
  (recall $(x, r) \prec (y, s)$ iff $d_\ell(x, y) < r - s$)
- Not Smyth-complete
- Not even d-algebraic Yoneda-complete: no d-finite element

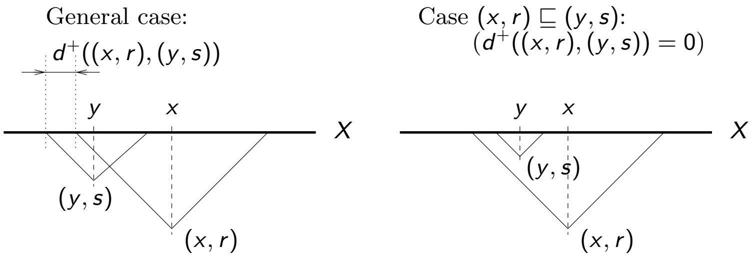
Outline

The Quasi-Metric Space of Formal Balls

Instead of considering $\mathbf{B}(X)$ as a poset, let us make it a quasi-metric space itself.

Definition (Rutten96)

Let $d^+((x,r),(y,s)) = \max(d(x,y) - r + s, 0)$

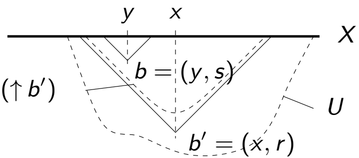


Note: $\sqsubseteq$ is merely the specialization quasi-ordering of d^+.

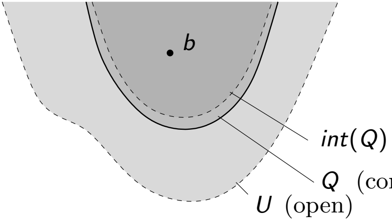
The C-Space of Formal Balls

Theorem

$\mathbf{B}(X)$ *(open ball topology) is a c-space, i.e., for all* $b \in U$ *open in* $\mathbf{B}(X)$, $b \in int(\uparrow b')$ *for some* $b' \in U$



Key: closed ball around (y, s), radius $\epsilon/2$, is $\uparrow(y, s + \epsilon/2)$

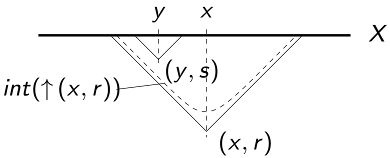


$\sim$ locally compact, where the interpolating compact is $\uparrow b'$ [Ershov73, Erné91]

The Abstract Basis of Formal Balls

Definition (Reminder)

Let $(x, r) \prec (y, s)$ in $\mathbf{B}(X)$
$\Leftrightarrow d(x, y) < r - s$
$\Leftrightarrow (y, s) \in int(\uparrow(x, r))$



Fact (Keimel)

c-space = abstract basis

Theorem

$\mathbf{B}(X), \prec$ *is an abstract basis, i.e.:*

- *(transitivity) if $a \prec b \prec c$ then $a \prec c$*
- *(interpolation) if $(a_i)_{i=1}^n \prec c$ then $(a_i)_{i=1}^n \prec b \prec c$ for some b*

C-Spaces and the Romaguera-Valero Thm (Pearl 6)

So $\mathbf{B}(X)$ is a c-space = an abstract basis

Note: sober c-space = continuous dcpo with way-below $\prec$

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Theorem (Romaguera-Valero10)

X Smyth-complete $\Leftrightarrow$ $\mathbf{B}(X)$ continuous dcpo with way-below $\prec$.

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Theorem (Romaguera-Valero10)

~~*X Smyth-complete $\Leftrightarrow$ $\mathbf{B}(X)$ continuous dcpo with way-below $\prec$.*~~

Theorem (JGL)

X Smyth-complete $\Leftrightarrow$ $\mathbf{B}(X)$ sober in its open ball topology.

Pearl 2 – Pearl 6 Crossover

In the same vein...

A monotone convergence space is one:

- that is a dcpo in its specialization order $\leq$
- whose topology is coarser that Scott

Every sober space is monotone convergence.

Theorem (Kostanek-Waszkiewicz10)

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~~*X Yoneda-complete* $\Leftrightarrow$ $\mathbf{B}(X)$ *dcpo.*~~

Theorem (JGL)

X Yoneda-complete $\Leftrightarrow$ $\mathbf{B}(X)$ *monotone convergence.*

Outline

Notions of Completion

Can we embed any quasi-metric space in a Yoneda/Smyth-complete one?

- Yes: Smyth-completion [Smyth88]
- Yes: Yoneda-completion [BvBR98]

Notions of Completion

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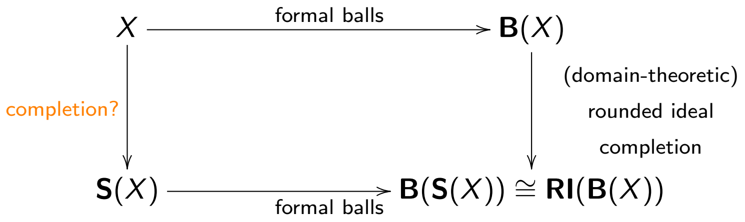
- Yes: Yoneda-completion [BvBR98]

Let us explore another way:



The Theory of Abstract Bases

A rounded ideal D in $B, \prec$ is a non-empty subset of B s.t.:

- (down closed) if $a \prec b \in D$ then $a \in D$
- (directed) if $(a_i)_{i=1}^n \in D$ then $(a_i)_{i=1}^n \prec b$ for some $b \in D$.

Theorem (Rounded Ideal Completion)

The poset $\mathbf{RI}(B, \prec)$ *of all rounded ideals, ordered by* $\subseteq$ *is a continuous dcpo, with basis* B*.*

Note: $\mathbf{RI}(\mathbf{B}(X), \prec)$ is just the sobrification of the c-space $\mathbf{B}(X)$.

The Formal Ball Completion

Definition

The formal ball completion $\mathbf{S}(X)$ is

- space of rounded ideals $D \in \mathbf{RI}(\mathbf{B}(X), \prec)$
  …with zero aperture $(\inf\{r \mid (x, r) \in D\} = 0)$
  $(\ldots = \text{Cauchy-weightedness})$
- with Hausdorff-Hoare quasi-metric

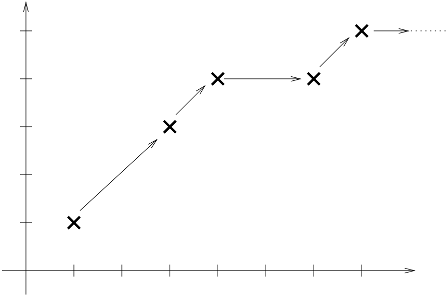
$$d^+_{\mathcal{H}}(D, D') = \sup_{(x,r)\in D} \inf_{(y,s)\in D'} d^+((x, r), (y, s))$$

Theorem

$\mathbf{B}(\mathbf{S}(X)) \cong \mathbf{RI}(\mathbf{B}(X))$

Proof. iso maps (D, r) to $D + r$ … as expected.

Comparison with Cauchy Completion

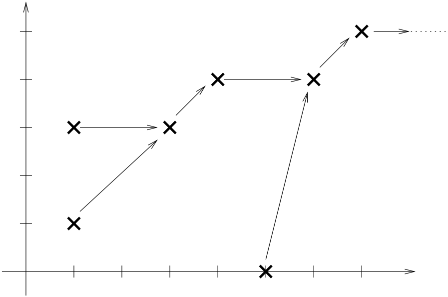


Imagine this is a chain $(x_i, r_i)_{i \in I}$ of formal balls

—

$(x_i)_{i \in I}$ is a Cauchy net

Comparison with Cauchy Completion



Imagine
this is a directed
family
$(x_i, r_i)_{i \in I}$
of formal balls

—

$(x_i)_{i \in I}$ is
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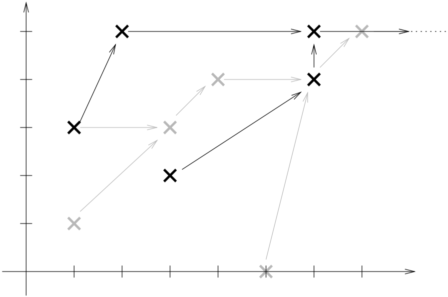
Comparison with Cauchy Completion

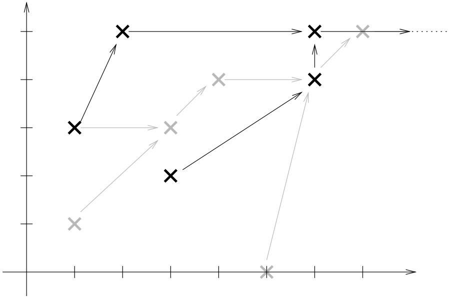


Now
here is another

—

with the same “limit”,
right?

Comparison with Cauchy Completion



Instead of quotienting,
(as in Smyth-completion)
take the union
of all these equivalent
directed families

This is a
rounded ideal.

Domain-Theoretic Properties

Theorem

$\mathbf{S}(X)$ *is a d-algebraic Yoneda-complete space*

not Smyth-complete in general

X *embeds into* $\mathbf{S}(X)$ *through* $\eta_X(x) = \{(y, r) \mid (y, r) \prec (x, 0)\}$

The d-finite elements of $\mathbf{S}(X)$ *are those in* X*.*

Similar to ideal completion $\mathbf{I}(X)$ of domain theory:

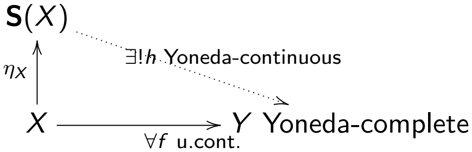
- $\mathbf{I}(X) = \mathbf{RI}(X, \leq)$
- $\mathbf{I}(X)$ is an algebraic dcpo
- X embeds into $\mathbf{I}(X)$
- The finite elements of $\mathbf{I}(X)$ are those in X

Universal Property

Theorem

$\mathbf{S}(X)$ *is the free Yoneda-complete space over* X.

I.e.,



Warning: morphisms:

q-metric spaces	uniformly continuous maps
Yoneda-compl. qms	u.c. + preserve d-limits ("Yoneda-continuity")
	(Yoneda-continuity=u.continuity in metric spaces)

Yoneda-Completion

- Let $[X \to \overline{\mathbb{R}}^+]_1 = \{$1-Lipschitz maps : $X \to \overline{\mathbb{R}}^+\}$, with sup quasi-metric $D(f, g) = \sup_{x \in X} d(f(x), g(x))$.
- Let $\eta^{\mathbf{Y}}_X(x) = d(\_, x) : X \to [X \to \overline{\mathbb{R}}^+]_1$

Definition (Yoneda completion [BvBR98])

$\mathbf{Y}(X) = D^{op}$-closure of $\mathrm{Im}(\eta^{\mathbf{Y}}_X)$ in $[X \to \overline{\mathbb{R}}^+]_1$

- Very natural from Lawvere's view of quasi-metric spaces as $\overline{\mathbb{R}}^{+op}$-enriched categories
  + adequate version of Yoneda Lemma
  (. . . , i.e., $\eta^{\mathbf{Y}}_X$ is an isometric embedding)
- $\mathbf{Y}(X)$ also yields the free Yoneda-complete space over X

Formal Ball and Yoneda Completion

S and **Y** both build free Yoneda-complete space

Corollary

$\mathbf{S}(X) \cong \mathbf{Y}(X)$, *naturally in* X

Concretely:

- $$\begin{aligned} D \in \mathbf{S}(X) &\mapsto \lambda y \in X \cdot \limsup_{(x,r)\in D} d(y,x) \\ &= \lambda y \in X \cdot \inf^{\downarrow}_{(x,r)\in D}(d(y,x)+r) \end{aligned}$$
- Inverse much harder to characterize concretely (unique extension of $\eta_X^{\mathbf{Y}} : X \to \mathbf{Y}(X) \ldots$)

Smyth-Completeness Again (Pearl 7)

- $\mathbf{S} \cong \mathbf{Y}$ is a monad on quasi-metric spaces
- but not idempotent $(\mathbf{S}^2(X) \not\cong \mathbf{S}(X)$, except if X metric)

Theorem (JGL)

Let X be quasi-metric. The following are equivalent:

- $\eta_X : X \to \mathbf{S}(X)$ *is bijective*
- $\eta_X : X \to \mathbf{S}(X)$ *is an isometry*
- X *is Smyth-complete*

Example: $X = \overline{\mathbb{R}}^+$ Y-complete, not S-complete, so $\mathbf{S}(\overline{\mathbb{R}}^+) \supsetneq \overline{\mathbb{R}}^+$
Example: any dcpo X, with $d_\leq(x, y) = 0$ iff $x \leq y$, is Yoneda-complete, but $\mathbf{S}(X)$ is ideal completion of X $(\neq X)$

Outline

Conclusion

- Pearl 1 [Wilson31]: countably-based $\Rightarrow$ hemi-metrizable
- Pearl 2 [Kostanek-Waszkiewicz10]: X Yoneda-complete iff $\mathbf{B}(X)$ dcpo, iff $\mathbf{B}(X)$ monotone convergence
- Pearl 3 [Romaguera-Valero10]: X Smyth-complete iff $\mathbf{B}(X)$ continuous dcpo with $\ll = \prec$
- Pearl 4 [Waszkiewicz10, Caristi]: self-maps f controlled by potential φ on Yoneda-complete space have fixed points
- Better than the open ball topology, the d-Scott topology
- Pearl 5 [Ali-Akbari et al. 10]: a Yoneda-complete space is Smyth-complete iff all its points are d-finite
- Pearl 6: X Smyth-complete iff $\mathbf{B}(X)$ sober
- $\mathbf{S}(X)$ through rounded-ideal completion, $\cong \mathbf{Y}(X)$
- Pearl 7: $\mathbf{S}(X)$ algebraic Yoneda-complete, but $X \cong \mathbf{S}(X)$ iff X Smyth-complete.

Formal Balls!